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Common Fixed Point Theorems in Partially Ordered Metric Spaces

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Abstract: *In this paper, common fixed point theorem for four mappings satisfying a contractive condition is obtained. This contractive condition is in the framework of partially ordered metric space. The four maps are partially weakly increasing. The theorem proved in this paper discusses a sufficient condition for the existence of a common fixed point for four maps. Our result generalizes the result of M. Abbas.*

Keywords: *Partially ordered set, partially ordered metric space, partially weakly increasing maps, dominating maps, annihilator.*

I. Introduction

Study of fixed point theorems in ordered metric spaces is a recent topic of research. Lattice is a structure equipped with an order. That is, two elements of a lattice can be compared.A. Tarski [1] invented fixed point theory in lattice and also gave some of its applications in 1955. A. Bjorner [2] published a paper on order preserving maps and unique fixed pointsin complete lattice in 1981. C. Blair et. al [3] extended the fixed point theorems in lattice structures. It is clear that a finite subset of ¡ with usual ordering between two numbers is a lattice. However the same set is a metric space with absolute value. Possibly, thisis the origin of ordered metric space. It is natural to ask about existence of fixed pointsin ordered metric space. Ran and Reurings [4] first established fixed point theorems inordered metric spaces. It was a natural extension of Banach contraction mapping principlein ordered metric space. Ran and Reurings [4] also mentioned the applications of their findings in matrix equations. We can see the further literature in Neito and Lopez [5]. Common fixed point theorems in ordered metric spaces were first investigated by M. Abbaset.al [6] in 2011. M. Abbas et. al presented a common fixed point theorem for four maps f, g, S and T in which the following condition was used.

$$d(Tx,Ty) \le \lambda M(x,y),$$

where

$$M(x,y) = \max\left\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{d(Sx,gy)+d(fx,Ty)}{2}\right\}$$

and $\lambda \in [0,1)$. The present paper also presents a common fixed point theorem in a partially ordered metric space. In this paper $M(x,y)$ is replaced by

$$M(x,y) = \max\left\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), d(Sx,gy), d(fx,Ty)\right\}$$

and $\lambda \in [0,1/2)$.

Thereafter study of common fixed point theory in ordered metric space has been the centre of vigorous research activity. We refer the further progress in this field in [15]-[21].The paper is organized as follows. In the next section there mentioned basic definitions and preliminaries required to state and prove main result. This section also includes some examples of the concepts defined and mentioned therein. This section is followed by the main result section.

II.Preliminaries and Definitions

Definition 2.1. [11] A partially ordered set (poset) is a non empty set X together with a binary relation denoted by $\le$ satisfying the following conditions

1) $x \le x$ for all $x \in X$,
2) $x \le y$ and $y \le x$ imply $x = y$ for all $x, y \in X$
3) $x \le y, y \le z$ imply $x \le z$ for all $x, y, z \in X$

The relation $\le$ is called a partial order on the set *X*. A partially ordered set is denoted by $(X,\le)$. Two elements x, y in a partially ordered set are said to comparable if either $x \le y$ or $y \le x$. An element $l \in X$ is said to be a least upper bound for the element $x, y \in X$ if $x \le l, y \le l$ and if there is an element $l' \in X$ such that $x \le l', y \le l'$, then $l \le l'$. An element $g \in X$ is said to be a greatest lower bound for the elements $x, y \in X$ if $g \le x, g \le y$ and if there is an element $g' \in X$ such that $g' \le x, g' \le y$, then $g' \le g$.

Definition 2.2. Let $(X,\le)$ be a partially ordered set and let $T : X \to X$ be a self map. Then the map T is said to be an order preserving map if $x \le y$ imply $Tx \le Ty$ and T is said to be order reversing map if $x \le y$ imply $Ty \le Tx$. A map T is called a monotone map if T is either order preserving or order reversing.

Definition 2.3. [1] Let $(X,\le)$ be a partially ordered set. Let f, g be two self maps on the set *X*. Then we define the following. A point $x \in X$ is called

1) A fixed of f if $fx = x$,
2) Coincidence point of the pair of mappings (f, g) if $fx = gx$,
3) Common fixed point of the pair of mappings (f, g) if $fx = gx = x$.

If x is a coincidence point of the pair of mappings (f, g), then $fx = gx = w$. The point w is called point of coincidence of the pair (f, g).

Definition 2.4. [19] The pair of mappings (f, g) is said to be weakly compatible if f and g commute at their coincidence point. That is $fx = gx$ imply $fgx = gfx$.

Definition 2.5. [4]Let $(X,\le)$ be a partially ordered set. Then the two self maps f, g on X are said to be weakly increasing if $fx \le gfx$ and $gx \le fgx$ hold for all $x \in X$.

Example2.6. Let $(X,\le)$ be a partially ordered set, where $X = [0,\infty)$ and $\le$ is usual ordering between two numbers. Define $f, g : X \to X$ by

$$f(x) = \begin{cases} x, \text{if } 0 \le x \le 1 \\ 0, \text{if } 1 < x \end{cases} \qquad g(x) = \begin{cases} \sqrt{x}, \text{if } 0 \le x \le 1 \\ 0, \quad \text{if } 1 < x \end{cases}$$

Then if $0 \le x \le 1$, gives $fx = x \le \sqrt{x} = gfx$ and $gx = \sqrt{x} = gfx$. If 1<*x*, then $fx = 0 = gfx$ and $gx = 0 = fgx$. Hence f, g are weakly increasing.

Example 2.7. Let $(X,\le)$ be a partially ordered set, where $X = \mathbb{R}^2$ and $\le$ is defined as follows:

for $(a_1,b_1),(a_2,b_2) \in \mathbb{R}^2$ define

$$(a_1,b_1) \le (a_2,b_2) \text{ if and only if } a_1 < a_2, \text{ and if } a_1 = a_2,\ b_1 \le b_2.$$

Define $f, g : X \to X$ by

$$f(a,b) = (\max(a,b), \min(a,b))$$

and

$$g(a,b) = \left(\max(a,b), \frac{a+b}{2} \right).$$

Then f, g are weakly increasing.

Definition 2.8. [1] Let $(X,\le)$ be a partially ordered set. Then the two self maps f, g on *X* are said to be partially weakly increasing if $fx \le gfx$ for all $x \in X$.

Example 2.9. Consider a partially ordered set $(X,\le)$, where $X = [0,1]$ and $\le$ is usual ordering between two numbers. Define $f, g : X \to X$ as follows:

$$fx = x^2, gx = \sqrt{x}$$

Then the pair (f, g) is partially weakly increasing. However, $gx = \sqrt{x} \not\le x = fgx$ for $x \in (0,1)$. So the pair (f, g) is not weakly increasing.

Definition 2.10. [1] Let $(X,\le)$ be a partially ordered set. Let f, g be two self maps on *X*. Then f is called a weak annihilator of g if $fgx \le x$ for all $x \in X$.

Example 2.11. Consider the partially ordered set $(X,\le)$, where $X = [0,1]$ and $\le$ is usual ordering between two numbers. Define $f, g : X \to X$ as follows:

$$fx = x^4, gx = x^6$$

Then

$$fgx = x^{24} \leq x$$

for all $x \in X$. Thus f is a weak annihilator of g .

Definition 2.12. [1] Let $(X,\leq)$ be a partially ordered set. A self map f on X is called a dominating map if $x \leq fx$ for all $x \in X$.

Example 2.13. Consider a partially ordered set $(X,\leq)$, where $X=[0,\infty)$ and $\leq$ as usual ordering between the numbers. Define $f: X \to X$ as

$$fx = \begin{cases} x^{\frac{1}{n}}, \text{if } 0 \leq x < 1 \\ x^{n}, \text{if } 1 \leq x < \infty \end{cases}$$

Then $x \leq fx$ for all $x \in [0,\infty)$. Hence f is a dominating map.

Definition 2.14. [11] Let (X,d) be a metric space. A partial order on this metric space can be defined as follows:

1) Associate a real number to each of the element of the set *X*.
2) If $x \in X$ is associated with the real number α and $y \in X$ is associated with the real number β , then define $x \leq y$ if and only if $d(x,y) \leq \beta - \alpha$.

We observe the following:

1) $x \leq x$ for all $x \in X$, because $d(x,x) = 0 = \alpha - \alpha$,
2) If $x \leq y$, then $d(x,y) \leq \beta - \alpha$. Also if $y \leq x$ then $d(y,x) \leq \alpha - \beta$. But as $d(x,y) = d(y,x)$, we must have $\alpha - \beta = 0$. Hence $\alpha = \beta$. Thus we obtain $x = y$.
3) Let $z \in X$ be associated with the real number γ . Let $x \leq y$ and $y \leq z$. Then we have $d(x,y) \leq \beta - \alpha$ and $d(y,z) \leq \gamma - \beta$. Thus we get
$$d(x,z) \leq d(x,y) + d(y,z) \leq (\beta - \alpha) + (\gamma - \beta) = \gamma - \alpha$$
Therefore $x \leq z$.

Hence the relation $\leq$ defines a partial order on the set *X*. The metric space (X,d) together with this partial order relation is called a partially ordered metric space. We denote a partially ordered metric space by $(X,\leq,d)$.

Ran and Reurings proved the following theorem in 2004 that extends Banach contraction mapping principle to partially ordered metric space.

Theorem 2.15. [18]Let $(X,\leq,d)$ be a partially ordered complete metric space such that for every $x, y \in X$ has a lower bound and has an upper bound. Let $T: X \to X$ be continuous, monotone map such that

1) $d(Tx,Ty) \leq \lambda d(x,y)$ for all $x, y \in X$ with $y \leq x$, where $\lambda \in [0,1)$ and
2) There exists $x_0 \in X$ such that $x_0 \leq Tx_0$ or $Tx_0 \leq x_0$.

Then T has a unique fixed point *z* in *X*. Moreover, for every $x \in X$, $\lim_{n\to\infty} T^n x = z$.

Neito and Lopez in 2005 proved the following result that removes continuity condition on the mapping T in the theorem 2.15.

Theorem 2.16. [15]Let $(X,\leq,d)$ be a partially ordered complete metric space. Suppose $T: X \to X$ be a monotone non decreasing mapping. Suppose that the following assertions hold:

1) $d(Tx,Ty) \leq \lambda d(x,y)$ for all $x, y \in X$ with $y \leq x$, where $\lambda \in [0,1)$,
2) There exists $x_0 \in X$ such that $x_0 \leq Tx_0$ and
3) If a non decreasing sequence $\{x_n\}$ converges to x in *X*, then $x_n \leq x$ for all n .

Then T has a unique fixed point in *X*.

Agarwal et. al generalized the above theorems by replacing $\lambda d(x,y)$ by a non decreasing function $\psi:[0,\infty) \to [0,\infty)$ with $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$ and proved the following theorem.

Theorem 2.17. [3] Let $(X,\leq,d)$ be a partially ordered metric space. Assume that there is a non decreasing function $\psi:[0,\infty) \to [0,\infty)$ with $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$ and suppose $T: X \to X$ be a non decreasing function with

$$d(Tx,Ty) \leq \psi(d(x,y))$$

For all $y \leq x$ in *X*. Also suppose that either T is continuous or if $\{x_n\}$ is a non decreasing sequence in *X* with limit $x \in X$, then $x_n \leq x$ for all n . If there is an $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point in *X*.

Remark 2.18. The non decreasing mapping $T : X \to X$ in the above theorem can be replaced by a non increasing mapping, provided $x_0 \leq Tx_0$ is replaced by $Tx_0 \leq x_0$.

Agarwal et. al [3] further generalized the above result by replacing the condition

$$d(Tx,Ty) \leq \psi(d(x,y))$$

by the condition

$$d(Tx,Ty) \leq \psi\left(\max\left(d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right)\right)$$

We refer further development in the fixed point theory of partially ordered metric space in [9,10,12,21]. I. Altunet. al [4] established fixed point theorems and common fixed point theorems in partially ordered metric spaces. Recently in 2011, M. Abbas [1] proved common fixed point theorems in partially ordered metric spaces.

Theorem 2.19. [1] Let $(X,\leq,d)$ be a partially ordered metric space. Let f,g,S,T be self maps on X, (f,T) and (g,S) be partially weakly increasing with $f(X) \subseteq T(X), g(X) \subseteq S(X)$ and dominating maps f,g be weak annihilators of T,S respectively. Also, for every two comparable elements $x, y \in X$,

$$d(fx,gy) \leq \lambda M(x,y)$$

where

$$M(x,y) = \max\left\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{d(Sx,gy)+d(fx,Ty)}{2}\right\}$$

for $\lambda \in [0,1)$ is satisfied. If one of $f(X), g(X), S(X), T(X)$ is a complete subspace of X, then (f,S) and (g,T) have unique common point of coincidence in X provided that for a non decreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all n and $y_n \to u$ implies $x_n \leq u$. Moreover, if (f,S) and (g,T) are weakly compatible, then f,g,S,T have a common fixed point.

III.Main Result

The condition in the theorem 2.19 is replaced by another suitable condition and the following theorem is proved.

Theorem 3.1. Let $(X,\leq,d)$ be a partially ordered metric space. Let f,g,S,T be self maps on X, (f,T) and (g,S) be partially weakly increasing with $f(X) \subseteq T(X), g(X) \subseteq S(X)$ and dominating maps f,g be weak annihilators of T,S respectively. Also, for every two comparable elements $x, y \in X$,

$$d(fx,gy) \leq \lambda M(x,y) \qquad (3.1)$$

where

$$M(x,y) = \max\left\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), d(Sx,gy), d(fx,Ty)\right\}$$

for $\lambda \in [0,1/2)$ is satisfied. If one of $f(X), g(X), S(X), T(X)$ is a complete subspace of X, then (f,S) and (g,T) have unique common point of coincidence in X provided that for a non decreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all n and $y_n \to u$ implies $x_n \leq u$. Moreover, if (f,S) and (g,T) are weakly compatible, then f,g,S,T have a common fixed point.

Proof: Let $x_0 \in X$ be any arbitrary point. Construct sequences $\{x_n\}$ and $\{y_n\}$ in Xsuch that

$$y_{2n-1} = fx_{2n-2} = Tx_{2n-1} \leq fTx_{2n-1}$$

and

$$y_{2n} = gx_{2n-1} = Sx_{2n} \leq gSx_{2n}$$

Since the dominating maps f and g are weak annihilators of T and S, respectively, so for all $n \geq 1$,

$$x_{2n-2} \leq fx_{2n-2} = Tx_{2n-1} \leq fTx_{2n-1} \leq x_{2n-1}$$

and

$$x_{2n-1} \leq gx_{2n-1} = Sx_{2n} \leq gSx_{2n} \leq x_{2n}$$

Thus, $x_n \leq x_{n+1}$ for all $n \geq 1$. Now (3.1) gives that

$$\begin{aligned} d(y_{2n+1}, y_{2n+2}) &= d(fx_{2n}, gx_{2n+1}) \\ &\leq \lambda M(x_{2n}, x_{2n+1}) \end{aligned}$$

for all n=1,2,3,…, where

$$
\begin{aligned}
&M(x_{2n}, x_{2n+1})\\
&= \max\{d(Sx_{2n}, Tx_{2n+1}), d(fx_{2n}, Sx_{2n}), d(gx_{2n+1}, Tx_{2n+1}), d(gx_{2n+1}, Sx_{2n}), d(fx_{2n}, Tx_{2n+1})\}\\
&= \max\{d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+2}), d(y_{2n+1}, y_{2n+1})\}\\
&= \max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+2})\}
\end{aligned}
$$

Now consider three cases:

Case 1. $M(x_{2n}, x_{2n+1}) = d(y_{2n}, y_{2n+1})$.

This case gives $d(y_{2n+1}, y_{2n+2}) \le \lambda d(y_{2n}, y_{2n+1})$. Hence, $d(y_n, y_{n+1}) \le \lambda d(y_{n-1}, y_n)$ for n=3,4,5,…. Therefore

$$
\begin{aligned}
d(y_n, y_{n+1}) &\le \lambda d(y_{n-1}, y_n)\\
&\le \lambda^2 d(y_{n-2}, y_{n-1})\\
&\dots\\
&\le \lambda^n d(y_0, y_1)
\end{aligned}
$$

For all $n \in ¥$. Then, for $m > n$,

$$
\begin{aligned}
d(y_n, y_m) &\le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \dots + d(y_{m-1}, y_m)\\
&\le \left(\lambda^n + \lambda^{n+1} + \dots + \lambda^m\right) d(y_0, y_1)\\
&\le \frac{\lambda^n}{1-\lambda} d(y_0, y_1)
\end{aligned}
$$

And so $d(y_n, y_m) \to 0$ as $n, m \to \infty$.

Case 2. $M(x_{2n}, x_{2n+1}) = d(y_{2n+1}, y_{2n+2})$.

This imply $d(y_{2n+1}, y_{2n+2}) \le \lambda d(y_{2n+1}, y_{2n+2})$, which gives $1 \le \lambda$. This is a contradiction.

Case 3. $M(x_{2n}, x_{2n+1}) = d(y_{2n}, y_{2n+2})$.

In this case

$$
\begin{aligned}
d(y_{2n+1}, y_{2n+2}) &\le \lambda d(y_{2n}, y_{2n+2})\\
&\le \lambda\left[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})\right]
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
&(1-\lambda) d(y_{2n+1}, y_{2n+2}) \le \lambda d(y_{2n}, y_{2n+1})\\
\Rightarrow \quad & d(y_{2n+1}, y_{2n+2}) \le \frac{\lambda}{1-\lambda} d(y_{2n}, y_{2n+1})\\
&= k d(y_{2n}, y_{2n+1})
\end{aligned}
$$

where $k = \dfrac{\lambda}{1-\lambda} < 1$. Thus we get,

$$
\begin{aligned}
d(y_{2n+1}, y_{2n+2}) &\le k d(y_{2n}, y_{2n+1})\\
&\le k^2 d(y_{2n-1}, y_{2n})\\
&\dots\\
&\le k^{2n+1} d(y_0, y_1)
\end{aligned}
$$

Letting $n \to \infty$ gives $\lim_{n\to\infty} d\left(y_{2n+1}, y_{2n+2}\right) = 0$. Hence $\lim_{n,m\to\infty} d\left(y_n, y_m\right) = 0$.

Hence case 2 is not possible whereas, case 1 and case 3 imply that the sequence $\{y_n\}$ is a Cauchy sequence in X. Now suppose that $S(X)$ is complete subspace of X . Then there exists u in $S(X)$, such that $Sx_{2n} = y_{2n} \to u$ as $n \to \infty$. Consequently, we find $v \in X$ such that $Sv = u$.

Now we claim that $fv = u$. Since, $x_{2n-2} \le x_{2n-1} \le gx_{2n-1} = Sx_{2n}$ and $Sx_{2n} \to Sv$, so that $x_{2n-1} \le Sv$ and since, $Sv \le gSv$ and $gSv \le v$ implies $x_{2n-1} \le v$. Consider

$$
\begin{aligned}
d(fv, u) &\le d(fv, gx_{2n-1}) + d(gx_{2n-1}, u)\\
&\le \lambda M(v, x_{2n-1}) + d(gx_{2n-1}, u)
\end{aligned}
$$

where

$$
M(v, x_{2n-1}) = \max\{d(Sv, Tx_{2n-1}), d(Sv, fv), d(Tx_{2n-1}, gx_{2n-1}), d(Sv, gx_{2n-1}), d(Tx_{2n-1}, fv)\}
$$

for all $n \in ¥$. Now we consider the five cases.

Case	**Value of** $M(v,x_{2n-1})$	**The expression** $d(fv,u) \le \lambda M(v,x_{2n-1})+d(gx_{2n-1},u)$ **becomes**	**Taking limit as** $n\to\infty$ **the expression** $d(fv,u) \le \lambda M(v,x_{2n-1})+d(gx_{2n-1},u)$ **becomes**	**Conclusion**
1	$d(Sv,Tx_{2n-1})$	$d(fv,u) \le \lambda d(Sv,Tx_{2n-1})+d(gx_{2n-1},u)$	$d(fv,u)\le \lambda d(u,u)+d(u,u)=0$	$fv=u$
2	$d(Sv,fv)$	$d(fv,u) \le \lambda d(Sv,fv)+d(gx_{2n-1},u)$	$d(fv,u)\le \lambda d(u,fv)+d(u,u) = \lambda d(u,fv)$	$fv=u$ since, $\lambda<1/2$
3	$d(Tx_{2n-1},gx_{2n-1})$	$d(fv,u) \le \lambda d(Tx_{2n-1},gx_{2n-1})+d(gx_{2n-1},u)$	$d(fv,u)\le \lambda d(u,u)+d(u,u)=0$	$fv=u$
4	$d(Sv,gx_{2n-1})$	$d(fv,u) \le \lambda d(Sv,gx_{2n-1})+d(gx_{2n-1},u)$	$d(fv,u)\le \lambda d(u,u),+d(u,u)=0$	$fv=u$
5	$d(Tx_{2n-1},fv)$	$d(fv,u) \le \lambda d(Tx_{2n-1},fv)+d(gx_{2n-1},u)$	$d(fv,u)\le \lambda d(u,fv)+d(u,u) = \lambda d(u,fv)$	$fv=u$ since, $\lambda<1/2$

Therefore in all cases we have $fv=Sv=u$. Thus v is a unique coincidence point of f and S. Since $u\in f(X)\subseteq T(X)$, there exists $w\in X$ such that $Tw=u$. Now the following argument show $gw=u$. As $x_{2n-1}\le x_{2n}\le fx_{2n}=Tx_{2n+1}$ and $Tx_{2n+1}\to Tw$ andso $x_{2n}\le Tw$. Hence, $Tw\le fTw$ and $fTw\le w$ imply $x_{2n}\le w$. Now consider

$$\begin{aligned} d(gw,u) &\le d(gw,fx_{2n})+d(fx_{2n},u)\\ &= d(fx_{2n},gw)+d(fx_{2n},u)\\ &\le \lambda M(x_{2n},w)+d(fx_{2n},u)\end{aligned}$$

where,

$$M(x_{2n},w)=\max\{d(Sx_{2n},Tw),d(Sx_{2n},fx_{2n}),d(Tw,gw),d(Sx_{2n},gw),d(Tw,fx_{2n})\}$$

for all $n\in ¥$. Consider the following five cases.

Case	**Value of** $M(x_{2n},w)$	**The expression** $d(gw,u) \le \lambda M(x_{2n},w)+d(fx_{2n},u)$ **becomes**	**Taking limit as** $n\to\infty$ **the expression** $d(gw,u) \le \lambda M(x_{2n},w)+d(fx_{2n},u)$ **becomes**	**Conclusion**
1	$d(Sx_{2n},Tw)$	$d(fv,u) \le \lambda d(Sx_{2n},Tw)+d(gx_{2n-1},u)$	$d(gw,u)\le \lambda d(u,u)+d(u,u)=0$	$gw=u$
2	$d(Sx_{2n},fx_{2n})$	$d(fv,u) \le \lambda d(Sx_{2n},fx_{2n})+d(gx_{2n-1},u)$	$d(gw,u)\le \lambda d(u,u)+d(u,u)=0$	$gw=u$
3	$d(Tw,gw)$	$d(fv,u) \le \lambda d(Tw,gw)+d(gx_{2n-1},u)$	$d(gw,u)\le \lambda d(u,gw)+d(u,u) = \lambda d(u,gw)$	$gw=u$ since $\lambda<1/2$
4	$d(Sx_{2n},gw)$	$d(fv,u) \le \lambda d(Sx_{2n},gw)+d(gx_{2n-1},u)$	$d(gw,u)\le \lambda d(u,gw)+d(u,u) = \lambda d(u,gw)$	$gw=u$ since $\lambda<1/2$
5	$d(Tw,fx_{2n})$	$d(fv,u) \le \lambda d(Tw,fx_{2n})+d(gx_{2n-1},u)$	$d(gw,u)\le \lambda d(u,u)+d(u,u)=0$	$gw=u$

Thus in all cases $gw=u=Tw$. Hence, w is a unique coincidence point of T and g. Also, u is unique point of coincidence of T and g.

Thus (f,S) and (g,T) have unique point of coincidence in *X*.

Now suppose (f,S) and (g,T) are weakly compatible. Then $fu = fSv = Sfv - Su = w_1$ (say) and $gu = gTw = Tu = w_2$ (say). Now $d(w_1,w_2) = d(fu,gu) \le \lambda M(u,u)$, where

$$\begin{aligned} M(u,u) &= \max\{d(Su,Tu), d(Su, fu), d(Tu,gu), d(Su,gu), d(Tu, fu)\} \\ &= \max\{d(w_1,w_2), d(w_1,w_1), d(w_2,w_2), d(w_1,w_2), d(w_2,w_1)\} \\ &= d(w_1,w_2) \end{aligned}$$

Thus, $d(w_1,w_2) \le \lambda d(w_1,w_2) \Rightarrow w_1 = w_2$. Hence $fu = gu = Su = Tu$. Thus, u is a coincidence point of f,g,S,T. Now the following argument show $gu = u$. Since $v \le fv = u$, we have

$$d(u,gu) = d(fv,gu) \le \lambda M(v,u)$$

where

$$\begin{aligned} M(v,u) &= \max\{d(Sv,Tu), d(Sv, fv), d(Tu,gu), d(Sv,gu), d(Tu, fv)\} \\ &= \max\{d(u,gu), d(u,u), d(gu,gu), d(u,gu), d(gu,u)\} \\ &= d(u,gu) \end{aligned}$$

Thus $d(u,gu) \le \lambda d(u,gu)$. Hence $gu = u$. Similarly it can be shown that $fu = u$. Hence u is a common fixed point of f,g,S,T.

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